Directed percolation with a wall or edge

Per Fröjdh†§, Martin Howard‡ and Kent Bækgaard Lauritsen‡

† NORDITA, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark
‡ Center for Chaos and Turbulence Studies, The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

Abstract. We examine the effects of introducing a wall or edge into a directed percolation process. Scaling ansatze are presented for the density and survival probability of a cluster in these geometries, and we make the connection to surface critical phenomena and field theory. The results of previous numerical work for a wall can thus be interpreted in terms of surface exponents satisfying scaling relations generalizing those for ordinary directed percolation. New exponents for edge directed percolation are also introduced. They are calculated in mean-field theory and measured numerically in $2 + 1$ dimensions.

1. Introduction

The impact of boundaries on critical phenomena has been the focus of much research in recent years (for extensive reviews of surface critical phenomena see [1, 2]). In the presence of a boundary certain (surface) quantities no longer scale as they do in the bulk, but possess different exponents which are dependent on the boundary conditions. In the past, most research has focused on the effects of surfaces in equilibrium critical phenomena and far less attention has been paid to boundaries in dynamical systems, such as in directed percolation (DP). The DP universality class is thought to describe a variety of phase transitions from non-trivial active into absorbing states [3] in processes such as epidemics, chemical reactions [4, 5], catalysis [6], the contact process [7], and certain cellular automata [8, 9]. Since all of these physical systems contain boundaries, an understanding of surface effects is very important.

The microscopic rules for bulk (bond) DP in $d + 1$ dimensions are extremely simple; any site at time t may make a connection to any of its $2d$ nearest neighbours at time $t + 1$ with growth probability p. Below a threshold, $\Delta = p - p_c < 0$, such a process always dies, whereas for $\Delta > 0$ there is a finite probability of survival, where the system enters the so-called active state [4]. At the transition point, the system is critical and scales anisotropically, i.e. the correlation lengths in time ($\parallel$) and space ($\perp$) diverge with different exponents, $\xi_\parallel \sim |\Delta|^{-\nu_\parallel}$ and $\xi_\perp \sim |\Delta|^{-\nu_\perp}$, respectively.

Above the upper critical dimension, $d_c = 4$, these exponents can be calculated using a simple mean-field theory. However, for $d < d_c$, fluctuation effects become important, and hence the computation of the exponents becomes a much harder task. The principal analytic technique for this calculation employs the equivalence between DP and Reggeon field theory [10, 11]. Using renormalization group techniques, the exponents can then

§ Present address: Department of Physics, Stockholm University, Box 6730, S-11385 Stockholm, Sweden.

be computed perturbatively in an $\epsilon = d_c - d$ expansion. These analytic techniques are supplemented by simulations and series expansions, which mean that, for example, the bulk critical exponents for $d = 1$ are known rather accurately [12]. Nevertheless, an exact solution for DP remains an open, and extremely important, problem.

In this paper we will be exclusively interested in the effects of boundaries on DP clusters. In order to isolate their effects, it is convenient to consider a semi-infinite system, where the cluster grows from a seed close to the surface. Series expansions [13] and numerical simulations [14] in 1+1 dimensions indicate that the presence of the wall alters several exponents. In particular, the percolation probability (order parameter),

$$P_1(\Delta) \sim \Delta^{\beta_1} \qquad \Delta \geqslant 0 \tag{1}$$

scales with an exponent β_1 rather than the standard exponent β (the subscript '1' refers to the wall). However, the scaling properties of the correlation lengths (as given by $\nu_\parallel$ and $\nu_\perp$) are *not* altered. More surprising, however, is the appearance of an apparently integer exponent describing the mean lifetime of a finite cluster in the presence of a wall:

$$\langle t \rangle \sim |\Delta|^{-\tau_1}. \tag{2}$$

Here $\tau_1 = \nu_\parallel - \beta_1 = 1.0002 \pm 0.0003$ in $1 + 1$ dimensions and is conjectured to be exactly unity [13]. If true, this would be a remarkable result, since none of the other exponents for DP are known exactly, and one even lacks evidence for them to be rational numbers. Note, however, that this situation is very different from the case of compact DP, where no vacancies within a cluster are allowed. This model is relatively simple to solve and most of the exponents, including τ_1, are integers (see [15] and references therein).

The purpose of this paper is to analyse the above results in the context of surface critical phenomena. We first of all write down scaling ansatze for the survival probability and cluster density which take boundary effects into account. From this we are able to derive the behaviour of, for example, the cluster mass in terms of surface (and bulk) exponents. We emphasize that the new exponent β_1 also describes the scaling of activity on the wall, thus β_1 is a so-called *surface* exponent [1, 2]. We next consider the appropriate field theory for DP in a semi-infinite geometry. This theory was first analysed by Janssen *et al* [16], where the appropriate surface exponents were computed to first order in $\epsilon = d_c - d$ using renormalization group techniques. The introduction of an inactive wall results in the so-called ordinary surface transition at the bulk critical point, for which sites close to the wall are less likely to be active than those in the bulk. In this picture, it is clear that only certain exponents are altered; more precisely, the boundary introduces one new independent exponent, while all the bulk exponents remain unaffected. By placing a finite seed close to the boundary, however, the distribution functions of the emerging clusters become sensitive to the new exponent. Unfortunately, these field theoretic methods yield little insight into why τ_1 should equal unity in $1 + 1$ dimensions. Hence, we conclude that the apparently integer value for τ_1 must be a special property of DP in $1 + 1$ dimensions, inaccessible to perturbative expansions about d_c.

Finally, we extend our analysis by allowing the wall to have an edge with opening angle α. This leads to the introduction of new angle-dependent *edge* exponents which govern the properties of clusters started close to the edge. We solve the corresponding mean-field theory and also determine the exponents numerically in $2 + 1$ dimensions using computer simulations.

2. Wall analysis

In this section we shall discuss the exponents associated with the growth of DP clusters in the presence of a wall. Some of our analysis will be similar in spirit to that of Grassberger [17], who analysed the case of ordinary percolation in a semi-infinite system.

First of all, let us examine the effects of introducing a $(d-1)$-dimensional wall at $x_\perp = 0$ [$x = (x_\parallel, x_\perp)$] into a DP process. Note that the labels parallel ($\parallel$) and perpendicular ($\perp$) refer here to directions relative to the wall (and not relative to the time direction). Consider a cluster arising from a single seed located next to the wall at $t = 0$. The probability that an infinite cluster can be grown from this seed is given by the percolation probability (1) which scales as Δ^{β_1} for $\Delta > 0$. Furthermore the scaling of the activity in the active state at the wall, i.e. the probability that a surface point at a later time belongs to this infinite cluster, also scales as Δ^{β_1}. Thus β_1 is an (independent) surface exponent in analogy with surface critical phenomena for equilibrium statistical mechanics [1, 2] (more details can be found in section 3, where we will also discuss how the surface-activity scaling Δ^{β_1} crosses over to the bulk-activity scaling Δ^β in the active state).

For a given bulk universality class (such as that of DP), several surface universality classes are possible. In our case, the lattice has simply been cut off and hence there will be fewer active points close to the surface. This corresponds to the boundary condition for the so-called ordinary transition (for which $\beta_1 > \beta$). The survival probability (the probability that the cluster is still alive at time t) has the form

$$P_1(t, \Delta) = \Delta^{\beta_1} \psi_1(t/\xi_\parallel) \tag{3}$$

where the scaling function ψ_1 is constant for $t \gg \xi_\parallel$ [4]. Furthermore the presence of the wall leaves the correlation length exponents $\nu_\perp$ and $\nu_\parallel$ unaltered [2, 16]. The mean lifetime of finite clusters (2) follows from the lifetime distribution $-\mathrm{d}P_1/\mathrm{d}t$ [13], thus yielding the exponent

$$\tau_1 = \nu_\parallel - \beta_1. \tag{4}$$

However, for $\nu_\parallel < \beta_1$, the leading contribution to $\langle t \rangle$ will be a constant, such that the above scaling relation breaks down and is replaced by $\tau_1 = 0$. Note, however, if one instead considers a spacetime geometry where the wall direction departs from the time direction, then all the above quantities will, as usual, cross over to bulk scaling (see also [13]).

For the density ρ_1 at the point (x, t) of a cluster growing from a single seed located next to the wall we make the scaling ansatz

$$\rho_1(x, t, \Delta) = \Delta^{\beta_1 + \beta} f_1(x/\xi_\perp, t/\xi_\parallel) \tag{5}$$

where the cluster density is defined to be the coarse-grained average density of active points. This ansatz may be properly justified using the field theory analysis of section 3, however, a more intuitive justification of the prefactors may be given as follows. The first factor of Δ^{β_1} comes from the probability that an infinite cluster can be grown from the seed. The second factor of Δ^β arises from the bulk scaling of activity in the active state, i.e. the (conditional) probability that the point (x, t) belongs to the infinite cluster grown from the seed (see also [17]). The shape of the cluster is governed by the scaling function f_1. In (5) we have assumed that the density is measured at a finite angle ϑ away from the wall (where $\sin \vartheta = x_\perp/x$), and suppressed the ϑ-dependence of f_1. In contrast, if the density is measured along the wall, $\vartheta = 0$, then the appropriate ansatz reads

$$\rho_{11}(x, t, \Delta) = \Delta^{2\beta_1} f_{11}(x/\xi_\perp, t/\xi_\parallel) \tag{6}$$

as we pick up a factor Δ^{β_1} rather than Δ^β for the probability that (x, t) at the wall belongs to the infinite cluster grown from the seed. In $1 + 1$ dimensions, ϑ has, of course, no

meaning. Instead, we have a crossover to $\rho_{11}(t, \Delta) = \Delta^{2\beta_1} f_{11}(t/\xi_\parallel)$ close to the wall. We also remark that for a seed located a (finite) distance away from the wall, the expressions are more complicated, although the above scaling forms (5) and (6) are still applicable for large times after a crossover from the bulk scaling $\rho(x, t, \Delta) = \Delta^{2\beta} f(x/\xi_\perp, t/\xi_\parallel)$.

By integrating the cluster density (5) over space and time, we arrive at the average size of finite clusters grown from seeds on the wall,

$$\langle s \rangle \sim |\Delta|^{-\gamma_1} \tag{7}$$

such that

$$\nu_\parallel + d\nu_\perp = \beta_1 + \beta + \gamma_1. \tag{8}$$

Hence, the surface exponent γ_1 is related to the previously defined exponents via a scaling law that naturally generalizes the usual $(d + 1)$-dimensional hyperscaling relation

$$\nu_\parallel + d\nu_\perp = 2\beta + \gamma. \tag{9}$$

It was noted by the authors of [13] that relation (9) is not fulfilled when the exponents for the wall geometry are substituted. Within the context of surface critical phenomena this 'failing' of hyperscaling is perfectly natural, since (9) is only a relation for (the unaltered) bulk exponents. The results of previous numerical simulations with a wall [13, 14] are in fact in very good agreement with the modified hyperscaling relation (8). We also note in passing that another generalization of hyperscaling has recently been proposed, although in the rather different context of a (bulk) model with multiple absorbing states [18], where the exponent β' in the survival probability depends continuously on the density of the initial configuration. Such a generalization of hyperscaling might also apply to recent results in [19] for DP with different fractal seeds as initial conditions.

Besides integrating the density (5), we can also integrate the density on the wall (6) over the $(d - 1)$-dimensional wall and time. This integration yields the average (finite) cluster size on the wall,

$$\langle s_{\text{wall}} \rangle \sim |\Delta|^{-\gamma_{1,1}} \tag{10}$$

where

$$\nu_\parallel + (d - 1)\nu_\perp = 2\beta_1 + \gamma_{1,1}. \tag{11}$$

However, in higher dimensions ($d \approx 2$ being a marginal case) this relation is not fulfilled as it would predict a negative $\gamma_{1,1}$. For this case, $\gamma_{1,1} = 0$, reflecting a constant contribution to (10), cf the comment after (4).

The cluster density also contains information on the connectivity correlations, as it is proportional to the probability that the seed at the origin is connected to the point (x, t). At criticality, we obtain from (5) the power-law decay

$$\rho_1(x, t) = x^{-(\beta_1 + \beta)/\nu_\perp} \tilde{f}_1(t/x^z) \tag{12}$$

where $z = \nu_\parallel / \nu_\perp$ is the dynamical exponent. This is nothing but the critical surface–bulk correlation function with prefactor $x^{-(d+\eta_{1,0})}$, which defines the exponent $\eta_{1,0}$ describing the power-law decay of correlations between the surface and the bulk. Hence,

$$\beta_1 + \beta = \nu_\perp(d + \eta_{1,0}) \tag{13}$$

which generalizes the normal DP relation

$$2\beta = \nu_\perp(d + \eta) \tag{14}$$

with η the anomalous dimension in the bulk. Furthermore, by identifying (6) with the surface–surface correlation function, it follows that

$$2\beta_1 = \nu_\perp(d + \eta_{1,1}) \tag{15}$$

where $\eta_{1,1}$ is the anomalous surface dimension. As expected, $2\eta_{1,0} = \eta_{1,1} + \eta$.

3. Field-theoretical analysis

We now turn to the field-theoretic description of DP with a wall and its connections with the above scaling picture. The action appropriate for DP with a wall at $x_\perp = 0$ is given by [16]

$$S = S_{\text{bulk}} + S_{\text{surface}} \tag{16}$$

$$S_{\text{bulk}} = \int \mathrm{d}^d x \int \mathrm{d}t \ \left(\bar{\phi}[\partial_t - D\nabla^2 - \Delta]\phi + \tfrac{1}{2}u[\bar{\phi}\phi^2 - \bar{\phi}^2\phi] \right) \tag{17}$$

$$S_{\text{surface}} = -\int \mathrm{d}^{d-1}x \int \mathrm{d}t \ \Delta_s \bar{\phi}_s \phi_s. \tag{18}$$

Here S_{bulk} is simply the action from Reggeon field theory [10], where ϕ is the local activity, $\bar{\phi}$ is the response field, and where we have defined $\phi_s = \phi(x_\parallel, x_\perp = 0, t)$ and $\bar{\phi}_s = \bar{\phi}(x_\parallel, x_\perp = 0, t)$. The surface term in S_{surface} corresponds to the most relevant interaction consistent with the symmetries of the problem and which also respects the absorbing state condition. Alternatively we can rewrite the action S in the form of a Langevin-type equation for the local activity $\phi(x, t)$,

$$(\partial_t - D\nabla^2 - \Delta)\phi(x, t) + \tfrac{1}{2}u\phi(x, t)^2 + \eta(x, t) = 0 \tag{19}$$

$$\langle \eta(x, t) \rangle = 0 \qquad \langle \eta(x, t)\eta(x', t') \rangle = u\phi(x, t)\delta^d(x - x')\delta(t - t') \tag{20}$$

where $\eta(x, t)$ is a Gaussian noise term. The multiplicative factor $\phi(x, t)$ in the noise correlator reflects the fact that $\phi = 0$ is the absorbing state. The presence of the wall implies the boundary condition at $x_\perp = 0$ of $D\partial_{x_\perp}\phi|_s = -\Delta_s\phi_s$.

For the systematic analysis of DP below the upper critical dimension, the action (16)–(18) remains the more useful description. One can show that, in the limit $t \to \infty$, the system reaches a steady-state, where the order parameter $\langle \phi(x_\perp, \Delta) \rangle$ develops a profile in the direction away from the wall:

$$\langle \phi(x_\perp, \Delta) \rangle = \Delta^\beta \varphi(x_\perp/\xi_\perp) \tag{21}$$

where the exponent β describes the density in the bulk, $x_\perp \gg \xi_\perp$, for which the scaling function φ is constant. The angular brackets denote averaging with respect to the action (16)–(18). Close to the wall, however, the order parameter scales with a different exponent than β. This is in analogy with surface critical phenomena for equilibrium statistical mechanics. The new exponent is denoted by β_1, and for $\Delta > 0$ it governs the other limit of the scaling function φ, giving

$$\langle \phi(x_\perp, \Delta) \rangle \sim \Delta^{\beta_1} x_\perp^{(\beta_1 - \beta)/\nu_\perp} \qquad x_\perp \ll \xi_\perp. \tag{22}$$

It is now a standard procedure to derive the form of the correlation functions within the field theory. These expressions involve the same β-exponents as defined above in equations (21) and (22), and are identical to the scaling forms derived earlier in section 2 (with the exception of an additional inhomogeneous term which is present for the surface–surface correlation function [2]). This establishes that the β-exponents defined in the field theory above are indeed the same as the β-exponents used earlier in the scaling theory, which were defined in terms of a percolation probability, as in equation (1). Note that the vanishing of (22) in the limit $x_\perp \to 0$ is simply an artefact of the continuum analysis (on a lattice the density on the wall simply scales as Δ^{β_1}).

Turning now to other aspects of the field theory, it is also straightforward to show (to *all* orders in perturbation theory) that the correlation length exponents are everywhere unchanged by the wall—as are all the exponents in the *bulk* (see [2, 16]). Furthermore the surface exponent β_1 is the *only* new exponent introduced by the wall. The critical exponents

can be calculated in a perturbative ϵ expansion around the upper critical dimension $d_c = 4$. Hence, quoting from [16], we have (identical to the case of DP without a boundary)

$$\beta = 1 - \frac{\epsilon}{6} + \mathrm{O}(\epsilon^2) \qquad \nu_\| = 1 + \frac{\epsilon}{12} + \mathrm{O}(\epsilon^2) \qquad \nu_\perp = \frac{1}{2} + \frac{\epsilon}{16} + \mathrm{O}(\epsilon^2) \tag{23}$$

where $\epsilon = 4 - d$. These exponents are related via hyperscaling (9) to γ governing the divergence of the bulk susceptibility (average cluster size) and via (14) to η governing the decay of connectivity correlations at criticality. Furthermore an ϵ expansion calculation for the surface exponent β_1 yields [16]

$$\beta_1 = \frac{3}{2} - \frac{7\epsilon}{48} + \mathrm{O}(\epsilon^2). \tag{24}$$

From the field theory of [16], it is not hard to verify that (8) is the appropriate generalization of (9), relating β_1 to

$$\gamma_1 = \frac{1}{2} + \frac{7\epsilon}{48} + \mathrm{O}(\epsilon^2) \tag{25}$$

which in terms of the field theory describes the divergence of the *surface* susceptibility due to the application of an infinitesimal bulk field.

The above results are certainly consistent with the numerical work of [13, 14], where $\nu_\|$, $\nu_\perp$ were measured in the presence of a wall and found to be unchanged from their bulk values. The behaviour of β_1 in (24) is also in qualitative agreement with the available data. Numerically, however, the value of the exponent τ_1 was found to be extremely close to unity in $1 + 1$ dimensions. This is in contrast to the above series results, which give

$$\tau_1 = -\frac{1}{2} + \frac{11\epsilon}{48} + \mathrm{O}(\epsilon^2) \tag{26}$$

using the scaling relation (4). Note that the mean-field value of τ_1 appears to be negative. This is not in fact the case—following the discussion after (4), we have $\tau_1 = 0$ in high enough dimensions. Nevertheless from (26) we see that the puzzle of why τ_1 seems to equal unity in $1 + 1$ dimensions cannot be answered by perturbative expansions about $d_c = 4$. Therefore this feature would appear to be a special property of DP with a wall in $1 + 1$ dimensions. This conclusion is certainly in agreement with the $(2 + 1)$-dimensional simulations of [14].

4. Edge analysis

We next turn to the case of DP in an edge geometry, where the cluster is started on an edge. It has been known for some time that the presence of an edge introduces new exponents, independent of those associated with the bulk or with a surface (see [20] for a discussion in the context of equilibrium critical phenomena, or [17] in the context of percolation). However, such edge geometries have not yet (to our knowledge) been analysed for the case of DP.

Consider a system, where we allow the wall to have an edge with an angle α. Hence, the edge can be viewed as the $(d - 2)$-dimensional intersection of two $(d - 1)$-dimensional walls. By placing the seed next to this edge, the surface exponent β_1 is replaced by the edge exponent $\beta_2(\alpha)$ (where of course $\beta_1 = \beta_2(\pi)$). Following the same arguments as before, the survival probability for a cluster starting from the edge has the scaling form

$$P_2(t, \Delta) = \Delta^{\beta_2} \psi_2(t/\xi_\|) \tag{27}$$

where ψ_2 is constant for $t \gg \xi_\|$. In other words, the percolation probability scales as $P_2(\Delta) \sim \Delta^{\beta_2}$. Furthermore, we also have the new scaling ansatz for the cluster density

$$\rho_2(r, t, \Delta) = \Delta^{\beta_2 + \beta} f_2(r/\xi_\perp, t/\xi_\|) \tag{28}$$

where r is the radial coordinate in a system of spherical polar coordinates centred on the speed. This ansatz generalizes (5), i.e. it applies for directions away from the edge and the walls. By replacing β with β_1 or β_2, we obtain the corresponding results for the density along the wall or the edge, respectively. Moreover, in analogy with (7) and (8) for seeds on a wall, we obtain the average (finite) size $\langle s \rangle \sim |\Delta|^{-\gamma_2}$ of clusters grown from a seed next to an edge, by integrating (28) over space and time. This yields the relation

$$\nu_{\parallel} + d\nu_{\perp} = \beta_2 + \beta + \gamma_2. \tag{29}$$

Similarly, by integrating the corresponding wall density over the $(d-1)$-dimensional wall and time, we obtain the average size of cluster activity on the wall due to a seed at an edge, $\langle s_{\text{wall}} \rangle \sim |\Delta|^{-\gamma_{2,1}}$, with

$$\nu_{\parallel} + (d-1)\nu_{\perp} = \beta_2 + \beta_1 + \gamma_{2,1}. \tag{30}$$

Let us once more remark that scaling relations such as (29) and (30) are only valid as long as the predicted γ-exponents are non-negative. Our results indicate that γ_2 should be zero for small enough angles α in any dimension, and the same holds for $\gamma_{2,1}$ also for somewhat larger angles. In principle, we can also define an exponent for the average cluster size at the edge by $\langle s_{\text{edge}} \rangle \sim |\Delta|^{-\gamma_{2,2}}$, with $\nu_{\parallel} + (d-2)\nu_{\perp} = 2\beta_2 + \gamma_{2,2}$. However, after inspecting our numerical results in the next section, we conclude that $\gamma_{2,2}$ should always be zero, with the possible exception of α close to 2π in $d=2$. We also note that the wall geometry in section 2 is a special case, such that $\gamma_2(\pi) = \gamma_1$ and $\gamma_{2,1}(\pi) = \gamma_{1,1}$, whereas $\gamma_{2,2}$ strictly refers to the edge.

As before, it is also straightforward to identify the various cluster densities with correlation functions between different domains p and q. It follows that

$$\beta_p + \beta_q = \nu_{\perp}(d + \eta_{p,q}) \qquad 2\eta_{p,q} = (\eta_{p,p} + \eta_{q,q}) \tag{31}$$

with $p, q = 0$ (bulk), 1 (wall) or 2 (edge).

We now proceed to calculate the exponents of this geometry in mean-field theory. Much of this calculation can be taken over directly from [20] where a similar mean-field calculation was performed for the case of an Ising model in an edge geometry. The appropriate terms in the action (17), (18) yield mean-field equations with some resemblance to those of the Ising case. Nevertheless the presence of the time derivative and a different nonlinear term leads to some important modifications. However, if, at the mean-field level, we are interested in calculating either the equal-time two-point correlation function or the susceptibility then we can immediately take over the results from [20]:

$$\eta_{0,0} = \eta = 0 \qquad \eta_{1,1} = 2 \qquad \eta_{2,2} = 2\pi/\alpha \tag{32}$$

and

$$\gamma_2 = 1 - \pi/2\alpha. \tag{33}$$

Furthermore, the correlation exponents $\nu_{\parallel}$ and $\nu_{\perp}$ are again everywhere unchanged by the presence of the edge, and retain their usual bulk values.

However, a calculation of the density exponent β_2 requires an analysis of the nonlinear term. Hence this exponent will differ from that in the edge Ising model. In our case, using equation (19), the mean-field local activity for $\Delta > 0$ must obey the equation

$$D\nabla^2\phi + \Delta\phi - (u/2)\phi^2 = 0 \tag{34}$$

with the condition that $\phi \to 2\Delta/u$ in the bulk. Note that ϕ is the activity in the steady state and is translationally invariant along the edge direction. With $r_{\perp}$ the perpendicular distance to the edge, the solution has the following scaling form

$$\phi(r_{\perp}) = (\Delta/u)F(r_{\perp}/\xi_{\perp}, \alpha). \tag{35}$$

As $r_\perp \to 0$ the quadratic term in equation (34) can be neglected, hence we obtain the Ising result, with ϕ behaving as $r_\perp^{\eta_{2,2}/2}$. Using $\nu_\perp = \frac{1}{2}$ we then have $\phi \propto \Delta^{1+\pi/2\alpha}$, and hence

$$\beta_2 = 1 + \pi/2\alpha. \tag{36}$$

As a check, we note that this satisfies (29) and (31) at the upper critical dimension $d_c = 4$. Similarly, we naively obtain the mean-field values $\gamma_{2,2} = 2\gamma_{2,1} = -\pi/\alpha < 0$, which means that $\gamma_{2,2} = \gamma_{2,1} = 0$, as discussed above. Of course we could become more sophisticated and use a field-theoretic approach to calculate the fluctuation corrections to all these mean-field values in an ϵ expansion around $d_c = 4$. Nevertheless below the upper critical dimension, we can still expect the mean-field values, and their dependence on the angle α, to be qualitatively followed.

Let us also mention that these ideas can easily be generalized in $3 + 1$ dimensions and higher where one, for example, can consider a cluster originating from a seed at the intersection of three walls. The percolation probability and cluster densities will then scale with a new *corner* exponent β_3 satisfying scaling relations analogous to (29) and (31).

5. Simulations

In this section we report the results of simulations of edge DP in $2 + 1$ dimensions for opening angles of $\alpha = \pi/2$, $3\pi/4$, π and $5\pi/4$. We use $(2 + 1)$-dimensional bond DP on a bcc lattice where $p_c = 0.287\,338(3)$, and with bulk exponents $\beta = 0.584(5)$, $\nu_\parallel = 1.295(6)$, $\nu_\perp = 0.734(5)$, and the dynamic exponent $z = \nu_\parallel/\nu_\perp = 1.765(3)$ [21, 22]. In the simulations we start from one seed located on the edge (wall for $\alpha = \pi$) and grow the DP cluster. Typically we average over $100\,000$ clusters in order to reduce the error bars to a few percent.

We measure the average position of activity

$$\langle r^2 \rangle = \frac{1}{N(t, \Delta)} \int dV \, r^2 \rho_2(r, t, \Delta) = t^{2/z} \, h(t\Delta^{\nu_\parallel}) \tag{37}$$

where r is the distance from the seed and the normalization quantity $N(t, \Delta)$ is the mass of the cluster at time t. Thus the average position yields the dynamic exponent $z = \nu_\parallel/\nu_\perp$. Our results show that z retains its bulk value in agreement with the theoretical prediction. Accordingly, we can use the bulk z value in our further analysis in order to obtain better estimates for the β_2-exponents.

Next, we measure the critical survival probability which has a power-law behaviour

$$P_2(t) \sim t^{-\beta_2/\nu_\parallel} \tag{38}$$

obtained from equation (27). The same power law also describes the number of active sites (for surviving clusters) on the edge at criticality as a function of time. We also measure the probability of having a cluster of mass s which has the critical scaling behaviour

$$p(s) \sim s^{-\tau_s} \tag{39}$$

where $\tau_s = 1 + \beta_2/(\nu_\parallel + d\nu_\perp - \beta)$ (cf [14]). In addition, we measure the average number of active sites at criticality (averaged over all clusters) as a function of time by integrating (28) over space,

$$N(t) \sim t^{d/z - \beta/\nu_\parallel - \beta_2/\nu_\parallel}. \tag{40}$$

However, if we average only over clusters which survive to infinity, then we have instead

$$N_{\text{surv}}(t) \sim t^{d/z - \beta/\nu_\parallel}. \tag{41}$$

By measuring the above quantities we can extract various estimates for the ratio $\beta_2/\nu_\parallel$ which eventually lead to the estimates for β_2 listed in table 1. In tables 2 and 3 we list our

Table 1. Estimates for the β_2 exponents for $(2+1)$-dimensional edge DP together with the mean-field values. The bulk and $(1+1)$-dimensional wall estimates [13, 14] are listed for reference. The mean-field value $\beta_2^{(MF)}$ is obtained from equation (36). Recall that $\beta_2(\pi) = \beta_1$.

Angle (α)	$\pi/2$	$3\pi/4$	π	$5\pi/4$	bulk
$\beta_1^{(1+1)}$			0.7338 ± 0.0001		0.2765 ± 0.0001
$\beta_2^{(2+1)}$	1.6 ± 0.1	1.23 ± 0.07	1.07 ± 0.05	0.98 ± 0.05	0.584 ± 0.005
$\beta_2^{(MF)}$	2	5/3	3/2	7/5	1

Table 2. Estimates for the τ_2 exponents for $(2+1)$-dimensional edge DP. The bulk and $(1+1)$-dimensional wall estimates [13, 14] are listed for reference. Note that $\tau_2^{(MF)} = 0$. Recall that $\tau_2(\pi) = \tau_1$.

Angle (α)	$\pi/2$	$3\pi/4$	π	$5\pi/4$	bulk
$\tau_1^{(1+1)}$			1.0002 ± 0.0003		1.4573 ± 0.0002
$\tau_2^{(2+1)}$	0.1 ± 0.05	0.20 ± 0.05	0.26 ± 0.02	0.38 ± 0.04	0.711 ± 0.007

Table 3. Estimates for the γ_2 exponents for $(2+1)$-dimensional edge DP together with the mean-field values. The bulk and $(1+1)$-dimensional wall estimates [13, 14] are listed for reference. The mean-field value $\gamma_2^{(MF)}$ is obtained from equation (33). Recall that $\gamma_2(\pi) = \gamma_1$.

Angle (α)	$\pi/2$	$3\pi/4$	π	$5\pi/4$	bulk
$\gamma_1^{(1+1)}$			1.8207 ± 0.0004		2.2777 ± 0.0001
$\gamma_2^{(2+1)}$	0.7 ± 0.1	1.0 ± 0.1	1.05 ± 0.02	1.20 ± 0.05	1.592 ± 0.009
$\gamma_2^{(MF)}$	0	1/3	1/2	3/5	1

estimates for τ_2 and γ_2. These quantities are obtained by measuring the average lifetime $\langle t \rangle$ and average size $\langle s \rangle$ for finite clusters for different values of Δ and then obtaining the exponents by carrying out a power-law fit. We observe that the results in $2+1$ dimensions qualitatively show the behaviour expected from the mean-field predictions. With one exception, we confirm that the scaling relation

$$\tau_2 = \nu_\parallel - \beta_2 \tag{42}$$

(cf the analogous expression (4) for a wall) and hyperscaling (29) are both fulfilled when error bars are taken into account. This exception occurs for the smallest angle where the relation (42) is not fulfilled. This is because as soon as β_2 becomes larger than $\nu_\parallel$ (as is the case for $\alpha = \pi/2$) the above relation breaks down, and instead the mean cluster lifetime becomes constant (i.e. $\tau_2 = 0$), cf the comment after (4). Using our results for β_2 we find that in $2+1$ dimensions τ_2 will reach its mean-field value of zero for an angle in between $\pi/2$ and $3\pi/4$. When τ_2 approaches zero the correction to scaling terms in the expression for $\langle t \rangle$ will affect the scaling making it difficult to obtain precise values.

6. Conclusions

We have analysed the impact of a wall or edge on a DP process in terms of surface critical phenomena. The presence of an inactive wall results in an *ordinary* phase transition between active and inactive surface states at the bulk critical point. A description of this transition

requires the introduction of one further independent exponent in addition to those present in the bulk. We have formulated a scaling ansatz for clusters grown near the surface, which incorporates these surface effects. We have also used the connection between DP and Reggeon field theory to justify our scaling ansatze and to examine the nature of the surface exponents. It turns out that the conjecture for $\tau_1 = 1$ in $1 + 1$ dimensions cannot be explained within the ϵ expansion. We also remark that it would be possible to examine other surface universality classes for evidence of rational exponents, particularly at the *special* transition. This transition occurs when the surface bond probabilities are enhanced such that not only the bulk, but also the surface, is at criticality. We note that the transition at this (multicritical) point requires the introduction of *two* new independent exponents. Lastly, we have for the first time analysed edge exponents in DP for edges with variable opening angles. We have derived the mean-field exponents and computed numerical values from computer simulations in $2 + 1$ dimensions.

Acknowledgments

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